ON THE ENERGY TRANSFER IN FPU LATTICES

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Abstract

We study the evolution of dynamics in the Fermi-Pasta-Ulam-α model in order to classify and characterize the behavior of the system. This classification is divided into three main time intervals, called *stages*, in which there is a qualitative change in the dynamics of the system. We compare every stage in the FPU dynamics with those of Toda's system. Thus we can use Toda as a tool to distinguish between the chaotic and integrable behavior in the FPU-α system.

Introduction

The one dimensional FPU-α lattice with fixed boundary conditions is described by the Hamiltonian

$$H^{FPU} = \frac{1}{2}\sum_{k=1}^{N-1} p_k^2 + \sum_{k=0}^{N-1}[\frac{1}{2}(q_{k+1}-q_k)^2 + \frac{\alpha}{3}(q_{k+1}-q_k)^3] \quad (1)$$

with $x(0) = x(N) = 0$.

The Toda lattice, described by the Hamiltonian function

$$H^T = \sum_{k=1}^{N-1} p_k^2 + \frac{1}{4a^2}\sum_{k=1}^{N-1} e^{2a(q_{k+1}-q_k)} - \frac{N}{4a^2} \quad (2)$$

can be regarded as an approximation of the FPU-α Hamiltonian (1) of order α^2, since

$$H^T = H^{FPU} - \sum_{i=1}^{N-1}\sum_{k=4}^{\infty} \frac{2^{k-2}a^{k-2}}{k!}(q_{i+1}-q_i)^k. \quad (3)$$

At the harmonic limit $\alpha = 0$ is

$$H^T = H^{FPU} = H_2 = \sum_{k=1}^{N-1} E_k \quad (4)$$

where E_k are the harmonic energies and $\omega_k = 2\sin\frac{k\pi}{2N}$ the harmonic frequencies for both systems.

In the present work, we excite the first normal mode $k = 1$ of the FPU-α and Toda systems, with initial conditions $q_i = A \cdot \sin\frac{\pi i}{N}$, $\dot{q}_i = 0$, for $i = 1, 2, ..., N-1$.

STAGE I

The harmonic energies of both systems are characterized by a sharp power law growth in time, due to acoustic resonances.

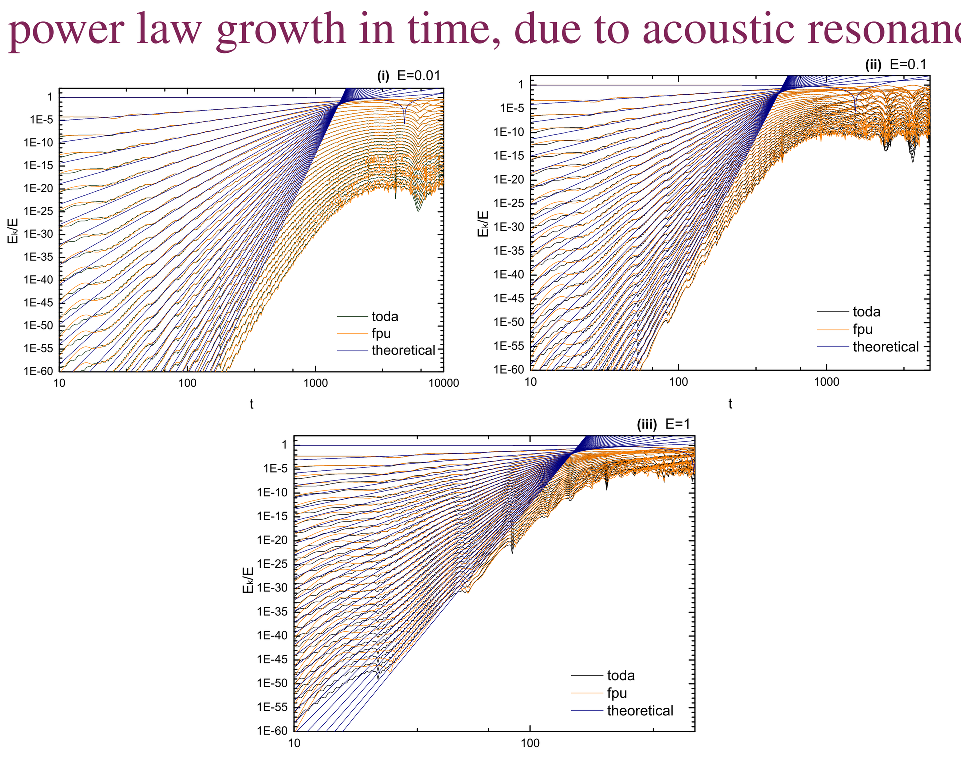

Figure 1: *FPU-α and Toda systems with $N = 32$, $\alpha = 0.33$. Normalized harmonic energy evolution plotted in logarithmic scale, for different values of the total energy: i) $E = 0.01$, ii) $E = 0.1$, iii) $E = 1$. Blue lines correspond to the power law $E_k \propto t^{2(k-1)}$.*

STAGE II

The harmonic energy spectrum saturates around a constant profile.

FPU-α and Toda systems exhibit in Fourier space the natural packet formation, in which the total energy is exchanged among few low-frequency modes [1], and the exponential energy localization of the *tail modes* [2], i.e. higher-frequency modes.

This stage holds for FPU-α up to the time that the energy spectra of tail modes slowly grow and the system tends to reach equipartition.

The exponential localization, of the averaged Toda energy spectra of the tail modes, are given by

$$Log(\frac{\overline{E}_k}{E}) = -\sigma k + \varrho \quad (5)$$

where

$$\sigma = \frac{eC_0}{N\sqrt{\alpha}\varepsilon^{1/4}} \quad (6)$$

and

$$\varrho = Log(\frac{C_1}{N\alpha\varepsilon^{1/2}}) \quad (7)$$

with $C_0 \simeq 3^{-1/2}$ and $C_1 \simeq 6$ (See Fig.4).

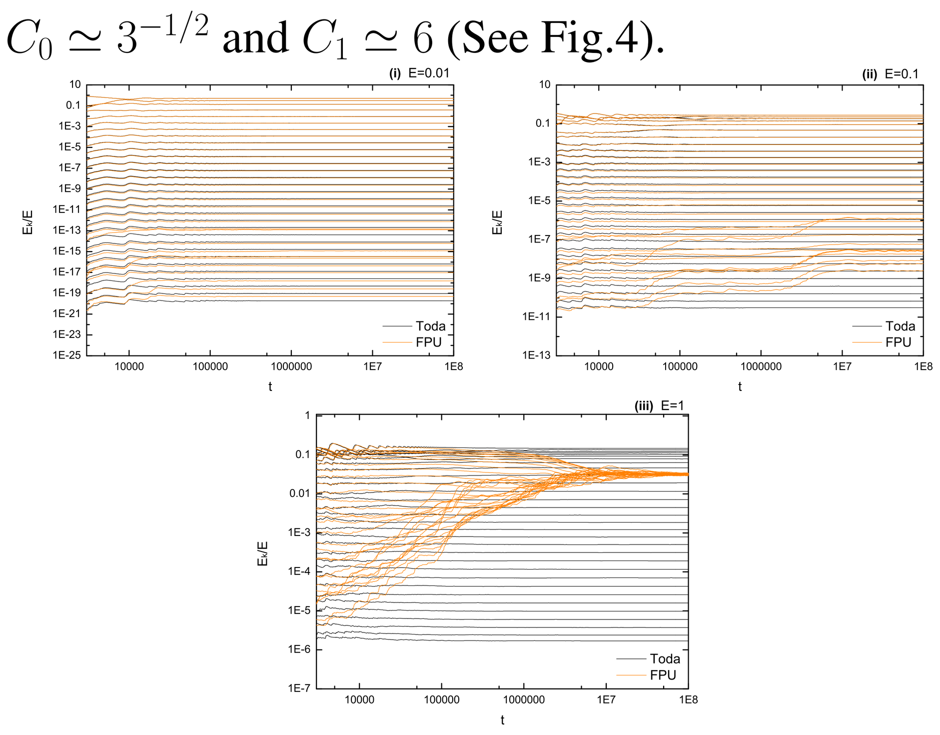

Figure 2: *FPU-α and Toda systems with $N = 32$, $\alpha = 0.33$. Normalized and time averaged harmonic energy evolution for both systems, plotted in logarithmic scale, for different values of the total energy: i) $E = 0.01$, ii) $E = 0.1$, iii) $E = 1$.*

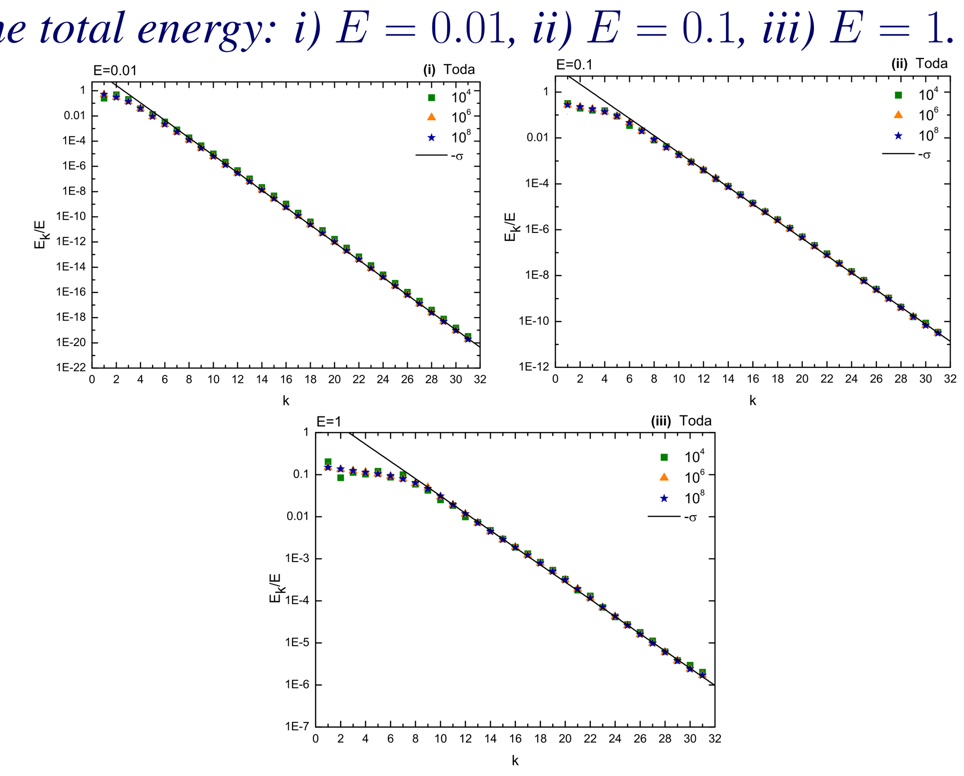

Figure 3: *Toda system with $N = 32$, $\alpha = 0.33$. Normalized and time averaged energy spectra for total energy: i) $E = 0.01$, ii) $E = 0.1$, iii) $E = 1$. In each panel we plot the spectra at times $10^4, 10^6, 10^8$. The black line corresponds to Eq. (5).*

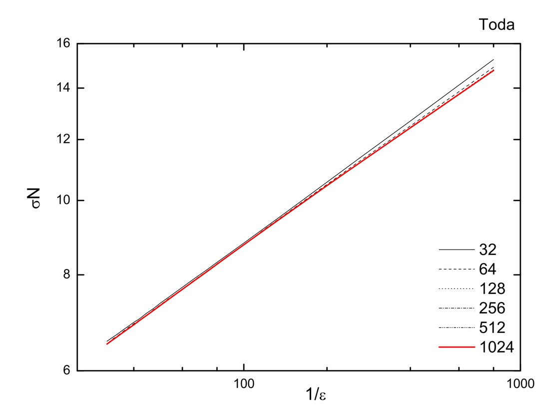

Figure 4: *Toda system with $\alpha = 0.33$. Numerical evidence for the validity of Eq. (6). The plot of $Log(\sigma N)$ versus $-Log(\varepsilon)$ is a line that saturates to Eq. (6) as the degrees of freedom N increase. The red line, which corresponds to $N = 1024$, is $-0.256Log(\varepsilon) + 0.4278$.*

STAGE III

Energy is diffused in FPU-α system from the packet to the tail.

We numerically estimate this diffusion by computing i) the moments of the normalized energy spectra, defined as

$$m_s = \sum_{k=1}^{N} k^s(\frac{E_k}{E})^s \quad (8)$$

ii) the sum of the harmonic energies of the last third part of the mode interval i.e. $[N/3, N]$, which we call *tail energy*

$$\eta = \sum_{k=N/3}^{N} \frac{E_k}{E}. \quad (9)$$

Numerical evaluations of both quantities show a power law increase in time, of the form

$$m_s \propto D_s t^{\gamma_s}, \;\; \eta \propto Dt^{\gamma} \quad (10)$$

for the moments and the tail energy respectively (see Figs.5 and 7).

By $\overline{m}_s$ and $\overline{\eta}$ we denote the moments and the tail energy, defined for the averaged harmonic energies $\overline{E}_k$.

The exponents of these powers laws are numerically evaluated and appear in Figs.6 and 7(ii). Both of them fluctuate very strongly indicating a dense region of invariant objects. As E increases, these objects are destroyed and the system tends to equipartition linearly in time ($\eta \propto t$).

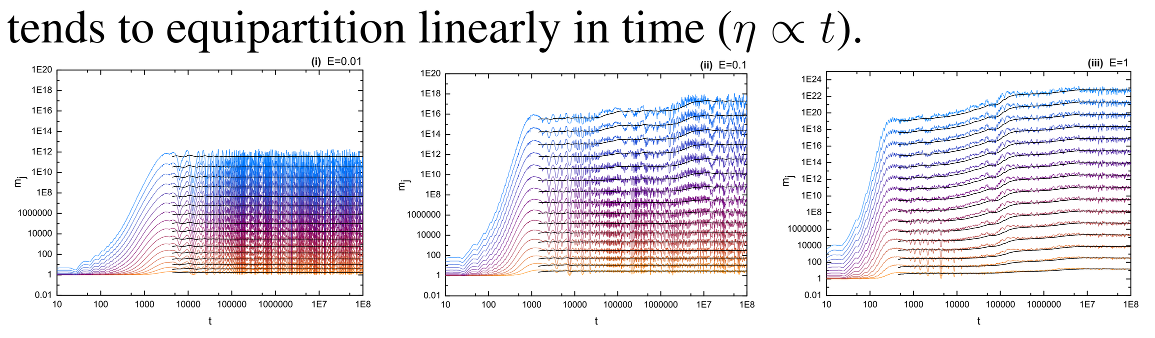

Figure 5: *FPU-α model with $N = 32$, $\alpha = 0.33$. Evolution of moments m_j, $j = 1, ..., 16$, plotted in logarithmic scale for total energy i) $E = 0.01$, ii) $E = 0.1$, iii) $E = 1$. The variation of moments, from m_1 to m_{16}, is shown by the line colour, which ranges from yellow to blue. The black curves correspond to moments $\overline{m}_j, j = 1, ..., 16$.*

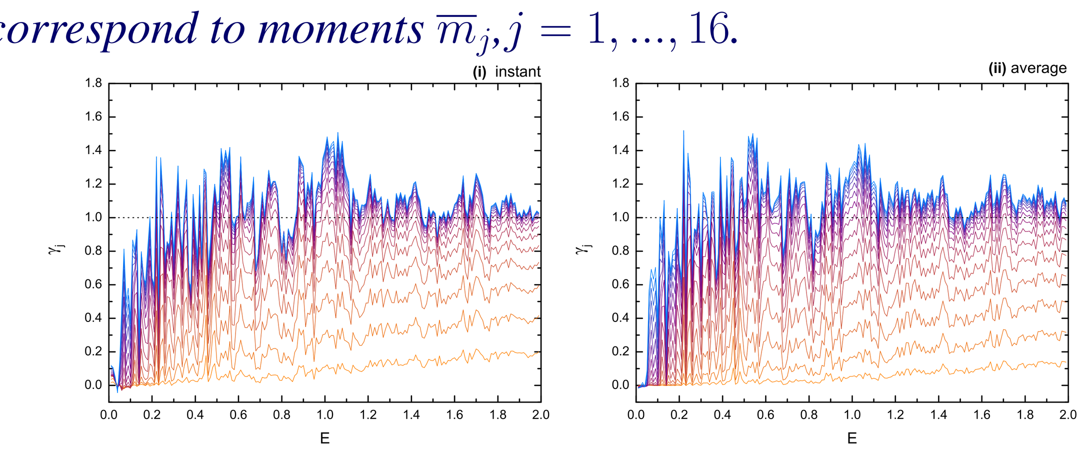

Figure 6: *FPU-α model with $N = 32$, $\alpha = 0.33$. i) The slope γ_j of the least squares linear fit, of moments m_j, $j = 1, ..., 16$ versus the total energy of the system E. Same line colours with Fig.5 are used. ii) The same for the slope $\overline{\gamma}_j$ of moments $\overline{m}_j$, $j = 1, ..., 16$.*

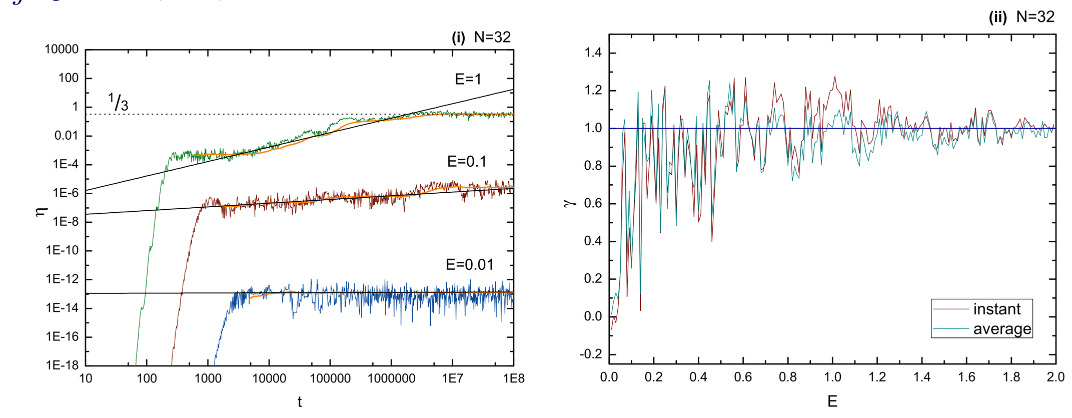

Figure 7: *FPU-α model with $N = 32$, $\alpha = 0.33$. i) Evolution of the tail energies, instantaneous η and averaged $\overline{\eta}$ (orange curves). The black straight lines correspond to the linear fit of $\overline{\eta}$ in the time windows $[10^4, 3 \cdot 10^6]$ for the total energies $E = 0.01$, $E = 0.1$ and $[10^3, 3 \cdot 10^5]$ for $E = 1$. Equipartition of the system is reached when $\eta = 1/3$. ii) The slope γ of the linear fit of η and $\overline{\eta}$ versus E.*

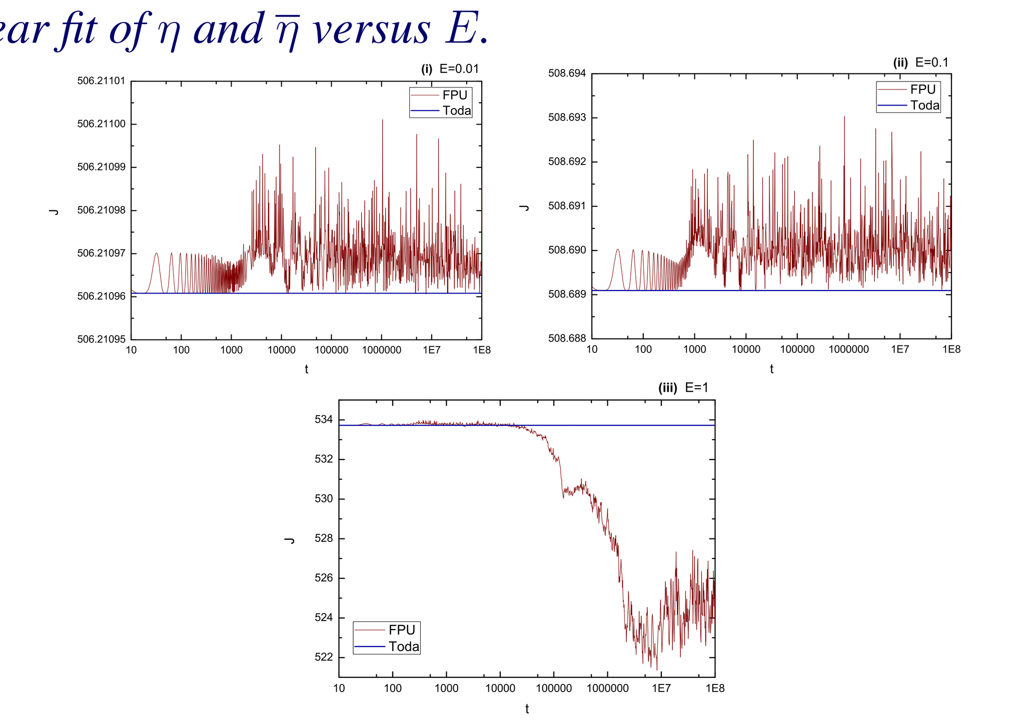

Figure 8: *Numerical computation of the 2^{nd} Toda integral J for both systems, for the total energies i) $E = 0.01$, ii) $E = 0.1$, iii) $E = 1$.*

Conclusions

The energy transfer in the FPU-α model from the lower frequency modes to the tail modes is initially very sharp, after that stops for a certain time window [3] and then starts again with a linear in time process, that leads the system to equipartition. Comparison with the Toda model shows that only the last part is due to non-integrability of FPU-α.

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References

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